Mathematics 222B Lecture 5 Notes

Daniel Raban

February 1, 2022

1 Sobolev Inequalities

1.1 The Gagliardo-Nirenberg-Sobolev inequality

We have been discussing Sobolev inequalities. Last time, we stated the following theorem.

Theorem 1.1 (Gagliardo-Nirenberg-Sobolev inequality). *Let $d \geq 2$. For $u \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\|u\|_{L^{\frac{d}{d-1}}(\mathbb{R}^d)} \leq \|Du\|_{L^1(\mathbb{R}^d)}.$$

To approach this, we proved a lemma:

Lemma 1.1 (Loomis-Whitney inequality). *Let $d \geq 2$. For $j = 1, \dots, d$, suppose $f_j = f_j(x^1, \dots, \widehat{x^j}, \dots, x^d)$. Then*

$$\left\| \prod_{j=1}^d f_j \right\|_{L^1(\mathbb{R}^d)} \leq \prod_{j=1}^d \|f_j\|_{L^{d-1}(\mathbb{R}^{d-1})}.$$

This answers the geometric question of controlling the measure of a set in $\mathbb{R}^d$ using the measure of its projections, by applying the lemma to $f_j = \mathbb{1}_{\pi_{x^j}(E)}$. Now let's prove the GNS inequality.

Proof. Observe that if we take a point $x \in \mathbb{R}^d$, then we can write

$$u(x) = \int_{-\infty}^{x^j} \partial_{x^j} u(x^1, \dots, x^{j-1}, y, x^{j+1}, \dots, x^d)\, dy,$$

using the fundamental theorem of calculus. Here, we use the compact support assumption to be sure this converges. This means that

$$|u(x)| \leq \int_{-\infty}^{x^j} |\partial_{x^j} u(x^1, \dots, x^{j-1}, y, x^{j+1}, \dots, x^d)| dy.$$

We can upper bound this by replacing x^j by ∞ and ∂_{x^j} by D:

$$|u(x)| \leq \underbrace{\int_{-\infty}^{\infty} |Du(x^1, \dots, x^{j-1}, y, x^{j+1}, \dots, x^d)| dy}_{\widetilde{f}_j(x^1, \dots, \widehat{x}^j, \dots, x^d)}.$$

This means that we have

$$|u(x)| \leq \left(\prod_{j=1}^{d} \widetilde{f}_j \right),$$

which we can write as

$$|u(x)|^{\frac{d}{d-1}} \leq \left(\prod_{j=1}^{d} \widetilde{f}_j^{\frac{1}{d-1}} \right),$$

Using the Loomis-Whitney inequality,

$$\begin{aligned}
\|u\|_{L^{\frac{d}{d-1}}}^{\frac{d}{d-1}} &= \int |u|^{\frac{d}{d-1}} \, dx \\
&\leq \int \prod_{j=1}^{d} f_j \, dx \\
&\leq \prod_{j=1}^{d} \|f_j\|_{L^{d-1}} \\
&= \prod_{j=1}^{d} \left(\int |f_j|^{d-1} \, dx^1 \cdots \widehat{dx^j} \cdots dx^d \right)^{\frac{1}{d-1}}
\end{aligned}$$

Observe that $|f_j|^{d-1} = \int_{-\infty}^{\infty} |Du(x^1, \dots, x^j, \dots, x^d)| \, dx^j = \int |Du| \, dx$, so

$$\leq \|Du\|_{L^1}^{\frac{d}{d-1}}. \qquad \square$$

Remark 1.1. GNS is the functional counterpart of the isoperimetric inequality. Given a function, we can make a layer cake decomposition in the y axis and apply the isoperimetric inequality to each part. This is useful for functions on manifolds where we have some geometric information.

1.2 Sobolev inequalities for L^p-based spaces with $p < d$

Now we will upgrade this to the case where we have other L^p spaces on the right hand side.

Theorem 1.2 (Sobolev inequalities for L^p-based spaces)**.** *Let $d \geq 2$, and assume that $1 < p < d$. For $u \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\|u\|_{L^q}(\mathbb{R}^d) \leq C\|Du\|_{L^p(\mathbb{R}^d)},$$

where $q = \frac{dp}{d-p}$.

What is q? We do dimensional analysis to figure out the exponent. On the left hand side, we have $[x]^{d/q}$, and on the right hand side, we have $[x]^{-1+d/p}$. If we solve for q, we get $q = \frac{dp}{d-p}$. This also gives us the restriction that $p < d$.

Proof. Take $v = |u|^{\widetilde{q}}$, where $\widetilde{q} = \frac{q}{d/(d-1)}$. Its derivative is $|Dv| = q|u|^{q-1}|Dv|$. This can be justified using approximation: approximate $|x|$ by $(\varepsilon^2 + x^2)^{1/2}$v and let $\varepsilon \to 0$. Then

$$\int |u|^{\widetilde{q}}\, dx = \int |v|^{\frac{d}{d-1}}\, dx$$

Using the GNS inequality,

$$\leq \left(\int |Dv|\, dx \right)^{\frac{d-1}{d}}.$$

It is at this point that we need the above approximation. But it works, using the dominated convergence theorem.

$$= \left(\int |u|^{\widetilde{q}-1} |Du|\, dx \right)^{\frac{d-1}{d}}$$

Using Hölder's inequality, we can put $|Du|$ into L^p, which puts $|u|^{\widetilde{}-1}$ in $L^{p'}$. By dimensional analysis, it must happen that

$$\leq \|u\|_{L^q}^{\frac{d-1}{d}(q-1)} \|Du\|_{L^p}^{\frac{d-1}{d}}.$$

This completes the proof. $\square$

Now we will upgrade this to every element in the abstract Sobolev space and to situations where we have a function which is bounded on an abstract domain.

Theorem 1.3. *Let $d \geq 2$, and assume that $1 \leq p < d$.*

(i) For $u \in W^{1,p}(\mathbb{R}^d)$,

$$\|u\|_{L^q}(\mathbb{R}^d) \leq C\|Du\|_{L^p(\mathbb{R}^d)},$$

where $q = \frac{dp}{d-p}$.

(ii) Let U be a bounded domain. For $u \in W_0^{1,p}(U)$,

$$\|u\|_{L^q}(U) \leq C\|Du\|_{L^p(U)},$$

where $q = \frac{dp}{d-p}$.

(iii) Let U be a bounded domain with C^1 boundary ∂U. Then for $u \in W^{1,p}(U)$,

$$\|u\|_{L^q}(U) \leq C\|Du\|_{W^{1,p}(U)},$$

where $q = \frac{dp}{d-p}$.

Proof.

(i) This is by density of $C_c^\infty(\mathbb{R}^d)$.

(ii) This is by density, as well.

(iii) This follows from extension and approximation. □

Remark 1.2. In (iii), we need both $\|u\|_{L^p}$ and $\|Du\|_{L^p}$ in the extension procedure. Compare this to the case (ii), where no information of u was needed, since "$u|_{\partial U} = 0$." By this reason, (ii) is called a **Poincaré inequality** or **Friedrich inequality**.

1.3 Sobolev inequalities for L^p-based spaces with $p > d$

Next, we investigate: What does $\|u\|_{W^{1,p}}$ tell us when $p \geq d$? This will be based on another way to relate u with its derivative, Du. Start with $u \in C^\infty(\mathbb{R}^d)$, and write down what we get by applying the fundamental theorem of calculus:

$$u(x) - u(y) = \int_0^1 = \frac{d}{ds}u(x + s(y-x))\,dx.$$

The key idea is to average to take advantage of the fact that we are in multiple dimensions. Take absolute values and average this in y: Fix $r > 0$, so

$$\frac{1}{|B_r(x)|}\int_{B_r} |u(x) - u(y)|\,dy \leq \frac{1}{|B_r(x)|}\int_{B_r(x)}\int_0^1 \left|\frac{d}{ds}u(x + s(y-x))\right|\,dx\,dy$$

By the chain rule, this derivative is $(y - x) \cdot Du(x + s(y-x))$.

$$\leq C\frac{1}{r^d}\int_{B_r(x)}\int_0^1 |x-y||Du(x + s(y-x))|\,dx\,dy$$

Let $\rho\omega = y - x$, so that $\rho = |y - x|$.

$$= C\frac{1}{r^d}\int_0^r\int_{\mathbb{S}^{d-1}}\int_0^1 \rho|Du(x + s\rho\omega)|\,ds\rho^{d-1}\,d\omega\,d\rho$$

Make another change of variables, so we can make $x + s\rho\omega$ into an actual radius and then evaluate on of the integrals. We do $t = s\rho$

$$= C\frac{1}{r^d}\int_0^r \int_{\mathbb{S}^{d-1}} \int_0^1 \frac{t^d}{s^d}\frac{1}{s}|Du(x+t\omega)|\,ds\,d\omega\,dt$$

Simplify the s integral and upper bound $t \leq r$:

$$\leq C\int_0^r \int_{\mathbb{S}^{d-1}} |Du(x+t\omega)|\,d\omega\,dt$$

$$= C\int_{B_r(x)} \frac{|Du|}{|x-y|^{d-1}}\,dy.$$

We can summarize this as a lemma:

Lemma 1.2. *Let $p > d$, let $d \geq 2$, and let $u \in C^\infty(\mathbb{R}^d)$. Then*

$$\frac{1}{|B_r(x)|}\int_{B_r} |u(x) - u(y)|\,dy \leq C\int_{B_r(x)} \frac{|Du|}{|x-y|^{d-1}}\,dy.$$

Theorem 1.4. *Let $p > d$ with $d \geq 2$, and take $u \in C^\infty(\mathbb{R}^d)$. Then*

$$|u(x) - u(y)| \leq C|x-y|^\alpha \|Du\|_{L^p(\mathbb{R}^d)},$$

where $\alpha = 1 - \frac{d}{p}$.

Again, we can find the value of α by dimensional analysis: $1 = \alpha + (-1) + \frac{d}{p}$ gives $\alpha = 1 - \frac{d}{p}$.

Proof. We will use the lemma. The idea is to introduce an auxiliary variable z and take the average over z on some domain U:

$$\frac{1}{|U|}\int_U |u(x)-u(y)|\,dz \leq \frac{1}{|U|}\int_U |u(x)-u(z)|\,dz + \frac{1}{|U|}\int_U |u(y)-u(y)|\,dz$$

Since $\frac{|B_r(x)|}{|U|} \simeq 1$,

$$\lesssim \frac{|B_r(x)|}{|U|}\int_{B_r(x)} |u(x)-u(z)|\,dz + \frac{|B_r(y)|}{|U|}\int_{B_r(y)} |u(y)-u(z)|\,dz$$

$$\lesssim \int_{B_r(x)} \frac{|Du|}{|x-z|^{d-1}}\,dz + \int_{B_r(y)} \frac{|Du|}{|y-z|^{d-1}}\,dz$$

$$\lesssim \|Du\|_{L^p} \left\| \frac{1}{|x-z|^{d-1}} \right\|_{L^{p'}(B_r(x))} + \|Du\|_{L^p} \left\| \frac{1}{|y-z|^{d-1}} \right\|_{L^{p'}(B_r(y))}$$

Now we just need to evaluate

$$\int_{B_r(0)} \frac{1}{|z|^{(d-1)p'}}\,dz \simeq r^\alpha. \qquad \square$$

1.4 Sobolev inequalities for L^p-based spaces with $p = d$

What about when $p = d$ (and $d \geq 2$)? In this case, the inequality $\|u\|_{L^\infty(U)} \leq \|u\|_{W^{1,d}(U)}$ fails.

Example 1.1. Here is a counterexample to the above inequality when $p = d = 2$. Take $U = B_1(0) \subseteq \mathbb{R}^2$ and

$$u(x) = \log\log\left(10 + \frac{1}{|x|}\right).$$

A popular remedy for $p = d$ is to think about bounded mean oscillation:

Definition 1.1. $u \in C^\infty$ has **bounded mean oscillation (BMO)** if

$$\|u\|_{\mathrm{BMO}} = \sup_{\substack{x \in \mathbb{R}^d \\ r > 0}} \frac{1}{|B_r(x)|} \int_{B_r(x)} \left| u(y) - \frac{1}{|B_r(x)|} \int_{B_r(x)} u \right| dy < \infty.$$

We can check that $\|u\|_{\mathrm{BMO}} \leq C\|Du\|_{L^d}$. We will discuss this next time and also introduce the concept of **Hölder space** to recontextualize the theorem we have just proven.